Superconnections and Grauert direct image theorem

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Let X be a smooth complex manifold. Denote its Dolbeault dg-algebra $(\Omega^{0,*}(X), \bar{\partial})$ by $\mathcal{A}(X)$. Denote by $\mathcal{P}_{\mathcal{A}(X)}$ the dg-subcategory of the dg-category of right $\mathcal{A}(X)$(modules, consisting of modules that, after forgetting the differential, become direct summands of finitely generated free graded $\mathcal{A}$-modules, so, modules of $(0,*)$-forms with values in some finite dimensional graded vector bundle. The objects of $\mathcal{P}_{\mathcal{A}(X)}$ are finite complexes of smooth complex vector bundles (E^*, d_0) together with the operations $d_i : E^n \to E^{n-i+1} \otimes_{C^\infty(X)} \mathcal{A}(X)$ which, after extending to the whole space $E^* \otimes_{C^\infty(X)} \mathcal{A}(X)$ by the Leibniz rule, satisfy $(\sum d_i)^2 = 0$. The dg-module condition implies that d_i for $i \neq 1$ are $C^\infty(X)$-linear operators, and operators d_1 are $(0,1)$-connections (not necessarily flat) on E^*. In [1] (see also [2]) and in [3] it is proven that for X compact, the homotopy category Ho$\mathcal{P}_{\mathcal{A}(\mathcal{X})}$ is equivalent to $D^b_{coh}(X)$, the derived category of bounded complexes of sheaves of $\mathcal{O}_X$-modules with coherent cohomology. Bondal and Rosly call the objects of $\mathcal{P}_{\mathcal{A}(X)}$ the $\bar{\partial}$-*superconnections*, and Block calls them *cohesive* modules. For a non-compact X, the complexes represented by objects of $\mathcal{P}_{\mathcal{A}(X)}$ form a strict subcategory of $D^b_{coh}(X)$. The goal of this note is to show how the existence of this enhancement allows one to obtain the proof of the Grauert's theorem on the coherence of higher direct images of coherent sheaves under proper morphisms, where all the functional analysis is hidden under the rug of the theory of elliptic operators. Although, this proof works only in the restricted case of a morhism between compact smooth manifolds.

First, note that any morphism $f : X \to Y$ decomposes into a composition of a closed embedding $i : X \to X \times Y$ and the projection $\pi : X \times Y \to Y$. The fact that the object $Ri_*\mathcal{F}$ of $D^b(Y)$ for $\mathcal{F}$ in $D^b_{coh}(X)$ has coherent cohomology follows from the Oka coherence theorem and from the Steinness of i. It seems highly nontrivial to construct a superconnection resolving $Ri_*\mathcal{F}$, see, however, [4], where this superconnection is constructed on the formal completion of a submanifold.

So, it suffices to consider the case of the projection π. Since coherence is a local property, the Grauert theorem for smooth compact manifolds would be a consequence of the following:

Theorem 1. *Let X and Y be smooth complex manifold, and X compact. Let $\mathcal{E}$ be an object of $\mathcal{P}_{\mathcal{A}}(X \times Y)$. Consider it as a complex of sheaves. Then for every point $y \in Y$ there exists an open $U \ni y$ in Y such that $\pi_*\mathcal{E}|_U$ is quasiisomorphic to an object in $\mathcal{P}_{\mathcal{A}}(U)$, where $\pi : X \times Y \to Y$ is the projection.*

Note that this indeed proves the Grauert theorem, as $\pi_*\mathcal{E}$ represents $R\pi_*\mathcal{E}$ since $\mathcal{E}$ is a bounded complex of fine sheaves.

Proof. The Dolbeault algebra of the product manifold splits as the tensor product (over functions) of the pullback of the Dolbeault algebras on the factors. Write

$$\mathcal{E} = E^* \otimes \mathcal{A}(X \times Y) = E^* \otimes_{C^\infty(X\times Y)} \pi_1^*\Omega_X^{0,*} \otimes_{C^\infty(X\times Y)} \pi_2^*\Omega_Y^{0,*}.$$

Define the $C^\infty(X \times Y)$-module F^n to be equal to $\oplus_{i+j=n} E^i \otimes_{C^\infty(X\times Y)} \pi_1^*\Omega_X^{0,*}$, and define the operators $f_i : F^n \otimes_{C^\infty(X\times Y)} \pi_2^*\Omega_Y^{0,m} \to F^{n-i+1} \otimes_{C^\infty(X\times Y)} \pi_2^*\Omega_Y^{0,m+n}$ by the requirement that $\sum f_i = \sum d_i$. The $\mathcal{A}(X \times Y)$-dg-module $(F^* \otimes_{C^\infty(X\times Y)} \pi_2^*\Omega_Y^{0,*}, \sum f_i)$ is evidently isomorphic to the dg-module $\mathcal{E}$, the only thing that changed is the grading. Denote this module by

$\mathcal{F}$. The $\mathcal{A}(Y)$-module $\pi_*\mathcal{F}$ is, after forgetting the differential, isomorphic to the module of $(0,*)$-forms on Y with values in an infinite-dimensional (but still of finite amplitude) graded vector bundles $\pi_{2*}F^*$. The modules of this kind are called *quasi-cohesive* in [5]; the definition there requires them to carry a topology with certain properties, this is irrelevant for now. We want to show that locally this is quasi-isomorphic to an object of $\mathcal{P}_{\mathcal{A}(X)}$.

We will do this with the help of spectral projectors and the homotopy perturbation theory. The fiber of $\pi_{2*}F^*$ over y is the complex of the global sections of a $\bar\partial_X$-superconnection $(F^*_y := \oplus\pi_{1*}E^* \otimes_{C^\infty(X)} \mathcal{A}(X), f_y = f^0|_{X\times\{y\}})$. Note that f_y itself decomposes into a sum $f_y = \sum_{i\geqslant 0} f^n_y : \pi_{1*}E^* \to \pi_{1*}E^{*-n+1} \otimes_{C^\infty(X)} \mathcal{A}^n(X)$. This is a complex of global sections of a complex of vector bundles over X. Since f_y is a $\bar\partial_X$-superconnection, its symbol coincides with the symbol of $\bar\partial_X$ and therefore the complex is elliptic. Choose arbitrarily a Hermitian metric on X and a Hermitian metric on E. Denote the corresponding formally adjoint operator as $f^\vee_y$ and the Laplacian $f_y f^\vee_y + f^\vee_y f_y$ by Δ_y. The Laplace operators are elliptic, self-adjoint and depend continuously on y. Their spectra $\operatorname{Spec}\Delta_y$ are discrete. Choose a positive number $\lambda \notin \operatorname{Spec}\Delta_y$ and denote by $\Pi^{\leqslant\lambda}_y$ the orthogonal projector onto a subspace of $\pi_{1*}F^*_y$ spanned by eigenforms with eigenvalues less than λ. The image of $\Pi^{\leqslant\lambda}_y$ is finite-dimensional. Denote the operator of inclusion of the image by $I^{\leqslant\lambda}_y$. Define by $G^{\leqslant\lambda}_y$ the λ-Green operator, which is zero on the image of $\Pi^{\leqslant\lambda}_y$ and is inverse to Δ_y on its orthogonal complement. The operator $H^{\leqslant\lambda}_y := -f^\vee_y G^{\leqslant\lambda}_y$ is the homotopy between the identity and $I^{\leqslant\lambda}_y\Pi^{\leqslant\lambda}_y$, meaning that $\operatorname{Id} - I^{\leqslant\lambda}_y\Pi^{\leqslant\lambda}_y = f_y H^{\leqslant\lambda}_y - H^{\leqslant\lambda}_y f_y$. Now, since Δ_y depends continuously on $y \in Y$, there exist an open set $U \ni y$ in Y such that for any $z \in U$ the number $\lambda \notin \operatorname{Spec}\Delta_z$. Moreover, from the integral formula for the spectral projection [6] one concludes that the projection $\Pi^{\leqslant\lambda}_z$ and the operators $I^{\leqslant\lambda}_z$ and $H^{\leqslant\lambda}_z$ vary continuously and therefore define a projection $\Pi^{\leqslant\lambda}_U$, an inclusion $I^{\leqslant\lambda}_U$ and a homotopy $H^{\leqslant\lambda}_U$ on a vector bundle $\pi_{2*}F^*|_U$, with the image of $\Pi^{\leqslant\lambda}_U$ being a finite dimensional vector bundle.

The appropriate version of the homotopy transfer theorem was described in [7]. Define the operator $f := \sum_{i\geqslant 1} f^i$, so $f + f_0$ is the differential on the $\mathcal{A}(\mathcal{Y})$-module $\pi_*\mathcal{F}$. Denote the operator $f(\sum_{i\geqslant 0}(H^{\leqslant\lambda}_U f)^i)$ by M. Note that this sum is actually finite, since $H^{\leqslant\lambda}_U$ preserves the degree in $\mathcal{A}(Y)$ and f strictly raises it. Then a straightforward computation shows that the operation $\delta := \Pi^{\leqslant\lambda}_U M I^{\leqslant\lambda}_U$ defines a $\bar\partial_Y$-superconnection on $\operatorname{Im}\Pi^{\leqslant\lambda}_U$. Moreover, the operator $H^{\leqslant\lambda}_U M I^{\leqslant\lambda}_U$ defines a quasi-isomorphism between $(\operatorname{Im}\Pi^{\leqslant\lambda}_U, \delta)$ and $\pi_*\mathcal{F}$ as $\mathcal{A}(Y)$-dg-modules, with the inverse quasi-isomorphism given by $\Pi^{\leqslant\lambda}_U M H^{\leqslant\lambda}_U$. The theorem 1 is proved.

One could wish to upgrade the construction above to an honest direct image functor $Rf_* : \mathcal{P}_{\mathcal{A}(X)} \to \mathcal{P}_{\mathcal{A}(Y)}$, at least for a projection map, or, more generally, for f a locally trivial fibration. To construct this, one needs to amend the category $\mathcal{P}_{\mathcal{A}(X)}$ to allow gluing objects of $\mathcal{P}_{\mathcal{A}(U)}$ given the appropriate glueing data. The appropriate description of the homotopy limits of dg-categories in terms of Čech cochains was described in [8], though the dg-categorical valued Čech cochains themselves were introduced in [9] under the name of twisted complexes. Second, one needs to make the choices of the spectral parameters functorial. A rather straightforward method to implement this was described in [10] under the name of enhanced operators, which are just operators together with a number not in their spectrum. The order on the real numbers defines a partial order on enhanced operators. While the introduction of these enhancements might, in principle, lead to a non-equivalent categories, the considerations of [10] show that these categories should have homotopy equivalent nerves, and hence enhancements do not change all the interesting categorical invariants. This upgrade is a topic for the future research.

The application the author has in mind are to the index theory. It seems plausible that the sufficient studying of the analytically flavoured dg models for derived categories of complex manifolds may lead, as in [11], to a proof of the variant of index theorems on the level of cochains, and that would make the appearence of η-invariants in them less mysterious. Another hint that these considerations are relevant to the problem is the fact that what obstructs $\pi_*\mathcal{E}$ to be represented by a superconnection is some sort of a spectral flow, which is known to be a K-theoretical invariant, see [12].

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