Two applications of homotopy transfer theorem

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Deformation quantizations

Let X be a complex analytic manifold with a holomorphic symplectic form $\omega \in \Omega^2(X)$. A **deformation quantization** on X is a sheaf $(\mathcal{A}, \star)$ of associative $\mathbb{C}[[\hbar]]$-algebras on X which is locally isomorphic to $\mathcal{O}[[\hbar]]$ as a sheaf of $\mathbb{C}[[\hbar]]$ modules, such that $\mathcal{A}/\hbar$ is locally isomorphic to $\mathcal{O}$ as the sheaf of algebras and

$$\frac{1}{\hbar}(f \star g - g \star f) = \omega(df, dg) \mod \hbar.$$

EXAMPLE: Let $X = \mathbb{C}^{2n}$ with the standart Darboux symplectic form. Then the **Weyl algebra**

$$\mathcal{O}[[\hbar]]/(z_i z_j - z_j z_i - \hbar\omega(dz_i, dz_j))$$

is a deformation quantization.

Deformation quantizations

QUESTION: To classify all deformation quantizations on a given (X, ω). Or at least to prove whether they exist or not.

IDEA: All symplectic forms **locally** look like the Darboux form. One can quantize locally and then try to glue.

A naive attempt to glue will fail, unless the transition functions are **affine**. In a coordinate-independent way:

THEOREM: (Bayen-Flato-Fronsdal-Lichnerowicz-Sternheimer) Suppose that X admits a **flat, torsion-free** connection ∇ such that $\nabla\omega = 0$. The sheaf $\mathcal{T}$ of **parallel** vector fields on X is a sheaf of (abelian) Lie algebras. The form ω defines a central extension of $\mathcal{T}$ called the **Heisenberg** Lie algebra $\mathfrak{h}$. Its universal enveloping algebra $U\mathfrak{h}$, suitably completed, is a deformation quantization of X.

Fedosov's idea

IDEA: (B. Fedosov) To make any torsion-free symplectic connection into a **flat** one (on a different bundle).

The **jet bundle** of any manifold is equipped with a canonical flat connection called the **Grothendieck connection**. A torsion-free symplectic connection gives a way to construct a **reduction** isomorphism between the jet bundle and the bundle $\hat{S}\Omega^1(X)$, the **completed symmetric algebra** of the cotangent bundle.

This idea allowed Fedosov to completely classify deformation quantizations of a **smooth (C^∞)** symplectic manifold M. The answer is given by

$$H^2_{dR}(M)[[\hbar]],$$

the power series in $\hbar$ with coefficients in the second de Rham cohomology of M.

Nest-Tsygan-Bezrukavnikov-Kaledin

Building on the idea of Fedosov, Nest-Tsygan and Bezrukavnikov-Kaledin were able to classify deformation quantizations for, respectively, smooth **complex analytic** manifolds and smooth **algebraic varieties**. In both cases, the answer is given by

$$\mathbb{H}^2(X, \Omega_{dR}^{\geq 1}(X))[[\hbar]],$$

the hypercohomology of truncated de Rham complex of X. In the complex analytic case this cohomology is isomorphic to the cohomology of the sheaf of **Hamiltonian** vector fields. This answer holds for varieties for which the natural map

$$\mathbb{H}^2(X, \Omega_{dR}^{\geq 1}(X)) \longrightarrow \mathbb{H}^2(X, \Omega_{dR}(X))$$

is an **injection**. This holds, for example, for Stein (resp., affine) manifolds and for compact Kähler (resp., smooth projective) manifolds.

Period map

All the theorems above are proved with the help of the so called **period map**:

$$\mathcal{P} : Quantizations \longrightarrow \mathbb{H}^2_{dR}(X)[[\hbar]].$$

In the C^∞ context this is an isomorphism. Generally, it is only an embedding. Its image is a **formal analytical** subset of $\mathbb{H}^2_{dR}(X)[[\hbar]]$.

PROBLEM: To describe the image.

This thesis grew out of desire to solve this problem and to describe the relation between the period map and the **Rozansky-Witten** invariants, noticed by Nest-Tsygan and Bezrukavnikov-Kaledin.

What is done in the thesis:

A Lie-theoretical description of the quantization problem: a **curved** L_∞-algebra $\mathcal{QUA}$ is constructed such that the set of deformation quantizations on a complex manifold X is the **Maurer-Cartan set** associated to it.

The period map is interpreted as the map of L_∞-algebras, thus reproving the results of Fedosov-Nest-Tsygan.

Rozansky-Witten invariants are more related to the L_∞-algebra $\mathcal{QUA}$ rather then to the period map: the **curvature** element of it is given by the RW-invariant associated to the formal linear combination of **theta-graphs**.

In the hyperKähler case, the formula for the period map is given, in terms of covariant derivatives of the curvature tensor and the Green operator, albeit complicated.

Lie theory

DEFINITION: A **curved dg-Lie algebra** is a graded Lie algebra L with a differential $d : L \longrightarrow L[1]$ and a curvature element $h \in L^2$ such that

$$d^2(x) = [h, x].$$

DEFINITION: The **Chevalley-Eilenberg coalgebra** $C(L)$ of a curved dg-Lie algebra L is a cofree cocommutative coalgebra Sym($L[1]$) with the differential

$$\begin{aligned} D(sv_1 \cdot \ldots \cdot sv_n) = \sum (-1)^{\varepsilon}(-1)^{|v_i|} s[v_i, v_j] \cdot sv_1 \cdot \ldots \cdot sv_n + \\ + \sum -(-1)^{|sv_1|+\ldots+|sv_{k-1}|} sv_1 \cdot \ldots \cdot sdv_k \cdot \ldots \cdot sv_n + \\ + sh \cdot sv_1 \cdot \ldots \cdot sv_n. \end{aligned}$$

Lie theory

DEFINITION: The **adjoint Lie algebra** TL of L is the dg-Lie algebra of coderivations of $C(L)$ with the differential $DA = [D, A]$.

DEFINITION: The **twist** of L by $a \in L^1$ is the curved dg-Lie algebra L^a which is L as a graded Lie algebra, with the differential $d^a(x) = d(x) + [a, x]$ and the curvature $h^a = d(x) + \frac{1}{2}[a, a] + h$.

The coalgebras $C(L)$ and $C(L^a)$ are, in general, not isomorphic, but their adjoint Lie algebras are, **via the map** e^{ad_a}:

$$e^{\mathrm{ad}_a} f(v_1, \ldots, v_n) = \sum \frac{1}{k!} f(a, \ldots, a, v_1, \ldots v_n).$$

DEFINITION: An L_∞**-map** between two curved dg-Lie algebras is a map $F : C(L) \longrightarrow C(M)$. Its **Taylor components** are restrictions f_i : $\mathrm{Sym}^i(L[1]) \longrightarrow M[1]$ for $i \geq \mathbf{0}$.

Profinite modules

DEFINITION: A topological k-module V is **profinite** if as it is isomorphic to the inverse limit of **finite dimensional vector spaces**.

REMARK: The functor of continious linear functionals $V \mapsto V^\vee$ gives an **equivalence** between the categories of profinite k-modules and k-modules.

The **topological tensor product** of a profinite module V and and a module A is given by $V \hat{\otimes} A := \mathrm{Hom}(V^\vee, A)$. The topological tensor product of two profinite modules V and W is given by $(V^\vee \otimes W^\vee)^\vee$. The symmetric algebra $\mathrm{Sym}(V^\vee)$ is isomorphic to the algebra of continuous symmetric polylinear functionals on V.

coLie theory

DEFINITION: A k-module V is a **Lie coalgebra** if $V^\vee$ is a profinite Lie algebra.

A Lie coalgebra $\mathfrak{c}$ has a Chevalley-Eilenberg **algebra** $C(\mathfrak{c})$.

The algebra $C(\mathfrak{c})$ is isomorphic to the algebra of continuous antisymmetric polylinear functionals on the profinite Lie algebra $\mathfrak{c}^\vee$.

Maurer-Cartan set

DEFINITION: A **Maurer-Cartan element** in a curved dg-Lie algebra L is an element $x \in L^1$ such that

$$dx + \frac{1}{2}[x,x] + h = 0.$$

For the next definition, suppose that L is flat over $\mathbb{C}[[\hbar]]$, $\hbar$-adically complete and has the property that $[L,L] \subset \hbar L$. Consider the curved dg-Lie algebra $L\hat{\otimes}\Omega^1 := \lim_n L/\hbar^n \otimes \mathbb{C}[t,dt]$.

DEFINITION: Two Maurer-Cartan elements x_0, x_1 are called **homotopy equivalent**, if there exists a Maurer-Cartan element $X(t) \in L\hat{\otimes}\Omega^1$ such that $X(0) = x_0$ and $X(1) = x_1$.

For an L_∞ map $F : C(L) \longrightarrow C(M)$ and a Maurer-Cartan element $x \in L^1$, the element

$$F(x) := \sum \frac{1}{n!} f_n(x,\dots,x),$$

provided this series converges, is a Maurer-Cartan element of M. If $x \sim y$, then $F(x) \sim F(y)$.

Maurer-Cartan set and deformations

The set of homotopy equivalence classes of a curved dg-Lie algebra L will be denoted by $\pi_0\mathcal{MC}(L)$. It is an **unpointed set, possibly empty**, functorial with respect to L_∞ maps.

EXAMPLE: The elements of $\pi_0\mathcal{MC}(\hbar TL[[\hbar]])$ are **$\hbar$-deformations** of L: curved dg-Lie algebras $\overline{L}$ over $k[[\hbar]]$ together with an isomorphism $\overline{L}/ \approx L[[\hbar]]$ of k-modules.

For a holomorphic symplectic manifold X we are going to describe a curved L_∞-algebra $\mathcal{QUA}(X)$ such that $\pi_0\mathcal{MC}(\mathcal{QUA}(X))$ is the set of isomorphism classes of deformation quantizations of X.

Twisting cochains

Let L be a curved dg-Lie algebra and let A be a commutative unital dg-algebra. Then the complex $L \otimes A$ inherits the curved dg-Lie algebra structure.

DEFINITION: A Maurer-Cartan element $\alpha \in L \otimes A$ is called a **twisting cochain** from L to A.

THEOREM: (Quillen): **In some cases** a twisting cochain from L to A is the same as the unital **dg-algebra homomorphism**

$$C^*(L) \longrightarrow A$$

from the dg-algebra $C^*(L) := \mathrm{Hom}_c(C(L), k)$ dual to $C(L)$ to A.

Quillen's theorem

THEOREM: (Quillen): If $\mathfrak{g}$ is a profinite Lie algebra, then a Maurer-Cartan element $\tau \in \mathfrak{g}\widehat{\otimes}A$ is the same as the unital **dg-algebra homomorphism**

$$C^*(\mathfrak{g}) \longrightarrow A$$

from the dg-algebra $C^*(\mathfrak{g}) := \mathrm{Hom}_c(C(\mathfrak{g}), k)$ **topologically dual** to $C(\mathfrak{g})$ to A.

PROOF: Let us write $\tau = \sum g_i \widehat{\otimes} a_i$. Then the corresponding homomorphism f maps $u \in \mathfrak{g}^\vee$ to $f(u) = \sum u(g_i)a_i$. Note that the sum converges since for any $u \in \mathfrak{g}^\vee$ only the finite number of elements $u(g_i)$ will be non-zero. Let us write $Du = \sum u_{(1)} \otimes u_{(2)}$. One then calculates that

$$f(Du) = (-\sum u_{(1)}(g_i)a_i)(-\sum u_{(2)}(g_j)a_j),$$

which is equal to

$$\sum_{i,j} u_{(1)}(g_i)u_{(2)}(g_j)a_i a_j = \sum \frac{1}{2} u([g_i, g_j])a_i a_j.$$

Quillen theorem

For

$$Du = \sum u_{(1)} \otimes u_{(2)}$$

we have that

$$f(Du) = (-\sum u_{(1)}(g_i)a_i)(-\sum u_{(2)}(g_j)a_j) = \\ = \frac{1}{2}\sum u([g_i, g_j])a_i a_j.$$

On the other hand, $df(u) = -\sum u(g_i)da_i$. The vanishing of the difference fof these quantities for any u means that

$$\frac{1}{2}\sum_{i,j}[g_i, g_j] \otimes a_i a_j + \sum_i g_i \otimes da_i = 0,$$

which is precisely the Maurer-Cartan equation for $\tau = \sum g_i \otimes a_i$.

Cones of Lie algebras

Let $\mathfrak{g}$ be an (ungraded) Lie algebra. Its **cone** is the dg-algebra $\mathrm{Co}\,\mathfrak{g}$ which is isomorphic to $\mathfrak{g}[-1] \oplus \mathfrak{g}$ with the differential induced by the identity map and the brackets induced by those in $\mathfrak{g}$.

A representation V of $\mathrm{Co}\,\mathfrak{g}$ is a complex together with operations i_v, L_v for $v \in \mathfrak{g}$ satisfying the Cartan identities:

$$[i_v, d] = L_v,\ \ [i_v, i_w] = 0,\ \ [L_v, L_w] = L_{[v,w]},\ \ [i_v, L_w] = i_{[v,w]}.$$

The subcomplex of $\mathrm{Co}\,\mathfrak{g}$-invariant vectors in V is called the **basic** subcomplex and is denoted by $\Gamma(V)$.

EXAMPLE: Let $P \longrightarrow X$ is a principal G-bundle. Then $\mathrm{Co}\,\mathfrak{g}$ acts on $\Omega^*(P)$ by vertical vector fields and

$$\Gamma(\Omega^*(P)) = \Omega^*(X).$$

Harish-Chandra pairs

DEFINITION: A **Harish-Chandra pair** is an dg-Lie algebra L, a Lie group G and a dg-Lie algebra morphism $\iota : \mathfrak{g} \longrightarrow L$ such that the corresponding representation of $\mathfrak{g}$ on L **integrates** to a representation of G.

The Chevalley-Eilenberg algebra $C^*(L)$ admits a Co $\mathfrak{g}$-action by derivations, with

$$i_v f(l_1, \ldots, l_n) = f(v, l_1, \ldots, l_n),$$

$$\begin{aligned} L_v f(l_1, \ldots, l_n) = (i_v d + d i_v) f(l_1, \ldots, l_n) = \\ = \pm f([\iota(v), l_i], l_1, \ldots, l_n) \\ \pm \sum f(\iota(v), [l_i, l_j], l_1, \ldots, l_n). \end{aligned}$$

Morphisms and connections

DEFINITION: An **infinity-morphism** between (L, G, ι) and (L', G', ι') is a smooth morphism of groups $F : G \longrightarrow G'$, and an L_∞-morphism f : $C^*(L') \longrightarrow C^*(L)$ such that f intertwines the actions of $\operatorname{Co}\mathfrak{g}$ and $\operatorname{Co}\mathfrak{g}'$.

Let A be a unital commutative dg-algebra with an action of $\operatorname{Co}\mathfrak{g}$ by derivations and let $(L, \mathfrak{g})$ be a Harish-Chandra pair with **profinite** L. A twisting cochain $\alpha \in \operatorname{Hom}^1(L^\vee, A)$ is called **$\operatorname{Co}\mathfrak{g}$-equivariant** if the corresponding morphism

$$C^*(L) \longrightarrow A$$

commutes with $\operatorname{Co}\mathfrak{g}$-actions.

Harish-Chandra torsors

DEFINITION: Let X be a smooth manifold and let (G, L) be a Harish-Chandra pair. A **Harish-Chandra torsor** over X is a

- G-torsor $P \longrightarrow X$,

- Co $\mathfrak{g}$-equivariant twisting cochain $\alpha \in \mathrm{Hom}^1(L^\vee, \Omega^*(P))$.

The map $C^*(L) \longrightarrow \Omega^*(P)$ induces a morphism

$$\Gamma(C^*(L)) = C^*_{cont}(L, G) \longrightarrow \Omega^*(X) = \Gamma(\Omega^*(P)).$$

It is called the **characteristic morphism**, or the **Gelfand-Fuks map**.

Harish-Chandra modules

DEFINITION: A **profinite** Harish-Chandra ∞-module over (G, L) is

- A profinite k-module V,
- A differential on $V \hat{\otimes} C^*(L) = \mathrm{Hom}_k(V^\vee, C^*(L)) =: C^*(L, V)$ making it into a dg-module over $C^*(L)$,
- An Co $\mathfrak{g}$-action on $\mathrm{Hom}_k(V^\vee, C^*(L))$ extending the action on $C^*(L)$.

EXAMPLE: The adjoint dg-Lie algebra $TL = \mathrm{Der}(C^*(L)) = \mathrm{Hom}(L^\vee, C^*(L))$.

DEFINITION: The **adjoint Lie algebra** of a Harish-Chandra pair (G, L) is $T(G, L) := \Gamma(TL)$.

Harish-Chandra deformations

DEFINITION: A **deformation** of a Harish-Chandra pair (G, L) is a pair $(G, \overline{L})$ with a morphism $(G, L) \longrightarrow (G, \overline{L})$ such that

- $\overline{L} \longrightarrow L$ is a deformation of L
- $G \longrightarrow G$ is the identity

THEOREM: The set $\pi_0 \mathcal{MC}(\hbar T(G, L)[[\hbar]])$ is in bijection with the set of isomorphism classes of deformations of (G, L).

Descent

DEFINITION: Let $(P, \alpha : C^*(L) \longrightarrow \Omega^*(P))$ be a (G, L)-torsor over X. The functor from profinite (G, L) ∞-modules to $\Omega^*(X)$-dg-modules given by

$$V \mapsto \Gamma(C^*(L, V)\widehat{\otimes}_{C^*(L)}\Omega^*(P)) =: \mathsf{desc}_{(P,\alpha)}(V)$$

is called the **descent functor**.

EXAMPLE: Descent of the adjoint module is the basic complex of twisted tensor product:

$$\mathsf{desc}\, L = \Gamma(L\widehat{\otimes}_\alpha\Omega^*(P))$$

The functor desc is **symmetric monoidal**, so it maps algebras in (G, L)-modules into algebras in $\Omega^*(X)$-modules.

Precomposition action

Let $(P, \alpha : C^*(L) \longrightarrow \Omega^*(P))$ be a (G, L)-torsor over X. We have the **extension of scalars** map

$$T(G, L) \longrightarrow T(\Gamma(L\widehat{\otimes}\Omega^*(P))$$

and the twisting isomorphism

$$e^{\mathrm{ad}_\alpha} : T(\Gamma(L\widehat{\otimes}\Omega^*(P)) \longrightarrow \Gamma(L\widehat{\otimes}_\alpha\Omega^*(P)).$$

Their composition

$$\mathrm{Prec} : T(G, L) \longrightarrow \Gamma(L\widehat{\otimes}_\alpha\Omega^*(P)) = \mathrm{desc}_{(P,\alpha)}(L)$$

is called the **precomposition action**.

The precomposition action allows to construct deformations of $\mathrm{desc}_{(P,\alpha)}(L)$ from deformations of (G, L).

Liftings of torsors

Let $(F, f) : (G, L) \longrightarrow (G', L')$ be a map of Harish-Chandra pairs. Let (P, α) be a (G, L)-torsor over X. Then

$$(P' := P \times_G G', \alpha' := f(\pi^* \alpha)),$$

where $\pi : P \longrightarrow P'$ is the projection, is a (G', L')-torsor over X. We will call the torsor (P', α') the **pushforward** of (P, α) along (F, f), and we will say that (P, α) is a **lifting**, or **reduction** of (P', α') to (G, L).

Two liftings are called **equivalent** if there is a gauge isomorphism between the corresponding torsors with connections that becomes identity after taking pushforward.

THEOREM: Let $\mu \in \hbar T(G, L)[[\hbar]]$ be a Maurer-Cartan element corresponding to the deformation $(G, \overline{L}) \longrightarrow (G, L)$. Let $\overline{\mathrm{desc}(L)}$ be the deformation of $\mathrm{desc}(L)$ corresponding to the element $\mathrm{Prec}(\mu)$. Then the set

$$\pi_0 \mathcal{MC}(\hbar \overline{\mathrm{desc}(L)})$$

is in bijection with the set of equivalence classes of **reductions of (P, α) to $(G, \overline{L})$**.

Liftings of torsors

The proof follows from an explicit description of $\overline{\text{desc}(L)}$, obtained from the unwinding of all the definitions involved. If $\text{desc}(L)$ is the basic subcomplex of

$$(L\hat{\otimes}\Omega^*(P),\ d+[\alpha,-]),$$

then $\overline{\text{desc}(L)}$ is the basic subcomplex of

$$(\overline{L}\hat{\otimes}\Omega^*(P),\ d+[\iota(\alpha),-],\ \frac{1}{2}[\iota(\alpha),\iota(\alpha)]),$$

where ι is the embedding

$$L = L\otimes 1 \longrightarrow L[[\hbar]] = \overline{L}.$$

Torsor of formal coordinate systems

Consider the algebra $\mathcal{A} := \mathbb{C}[[t_1, \dots, t_n]]$. Its automorphism group Aut $\mathcal{A}$ is naturally a projective limit of Lie groups — a prounipotent extension of $GL(n)$. The Lie algebra of Aut $\mathcal{A}$ is the Lie algebra $\mathrm{Der}_0\, \mathcal{A}$ of vector fields on a formal disk preserving the origin. It lies in the bigger Lie algebra of all derivations Der $\mathcal{A}$.

THEOREM: (Gelfand-Kazhdan) Any smooth complex manifold X is endowed with a functorial (Aut $\mathcal{A}$, Der $\mathcal{A}$)-torsor X_{coord}.

This torsor is the torsor of the trivializations of the **jet bundle** of X.

Jets

Let X be a complex manifold with transition functions g_{ij}. Taylor series of g_{ij} determine an $\mathrm{Aut}\,\mathcal{A}$-torsor X_{coord} over X. The associated algebra bundle

$$X_{coord} \times_{\mathrm{Aut}\,\mathcal{A}} \mathcal{A} \longrightarrow X$$

is called the **jet bundle** J of X.

The jet bundle is endowed with a natural holomorphic flat connection ∇_G, called the **Grothendieck connection**. In local coordinates it is given by

$$d - \sum dz_i \otimes \frac{\partial}{\partial t_i} = \sum dz_i \otimes (\frac{\partial}{\partial z_i} - \frac{\partial}{\partial t_i}).$$

Its sheaf of flat sections is the structure sheaf $\mathcal{O}_X$.

Jets

The jet construction could be performed in a coordinate-free language and with coefficients. Let E be a vector bundle on X. Then

$$J(E) := \lim_n \pi_{1*}(\mathcal{O}_{X\times X}/I^{n+1} \otimes_{\pi_2^{-1}\mathcal{O}_X} \pi_2^{-1}E),$$

where I is the ideal of the diagonal in $X \times X$. We have $J(\mathcal{O}) = J$.

The connection ∇_G in this construction is the de Rham differential along the first factor.

The sheaf of flat sections of $J(E)$ is the original sheaf E.

Transitivity

The solder form of a connection ∇_G is a form $\alpha \in \Omega^1_{X_{coord}} \hat{\otimes} \operatorname{Der} \mathcal{A}$ that in local coordinates on X_{coord} looks like

$$\sum -\frac{\partial}{\partial t_i} \otimes dz_i + \sum \iota(\tilde{v}_j) \otimes dv_j,$$

where $\tilde{v}_j$ foirm a basis of the Lie algebra $\operatorname{Der}^0 \mathcal{A}$, dv_j are the corresponding dual 1-forms and ι is the embedding $\operatorname{Der}^0 \mathcal{A} \longrightarrow \operatorname{Der} \mathcal{A}$.

One can interpret α as a morphism of bundles $\alpha^\vee : \operatorname{Der} \mathcal{A} \longrightarrow T_{X_{coord}}$. From the coordinate description one sees that α is **an isomorphism**.

DEFINITION: A (G, L)-Harish-Chandra torsor (P, α) over X is called **transitive** if the morphism $\alpha^\vee : L \longrightarrow T_P$ is an isomorphism.

THEOREM: (Beilinson-Drinfeld) The torsor X_{coord} is the **unique, up to a unique isomorphism** transitive $(\operatorname{Aut} \mathcal{A}, \operatorname{Der} \mathcal{A})$ torsor over X.

For a transitive (G, L)-torsor (P, α) over X, a point $p \in P$ over $x \in X$ induces an isomorphism between Spf $\hat{\mathcal{O}}_x$ and Spf $\mathbb{C}[[(L/\mathfrak{g})^\vee]]$.

Geometric structures

$(\mathrm{Aut}\,\mathcal{A}, \mathrm{Der}\,\mathcal{A})$-equivariant bundles and operators on a formal disc descend, with the help of X_{coord} to natural bundles over X. For example, the formal de Rham complex $\Omega^*(\mathrm{Spf}\ \mathcal{A})$ descends to the jet bundle of the de Rham complex $\Omega^*(X)$.

Suppose now that n is even and $\omega \in \Omega^2(\mathrm{Spf}\ \mathcal{A})$ is the standard symplectic form

$$\hat{\omega} = dt_1 \wedge dt_{n+1} + \ldots + dt_n \wedge dt_{2n}.$$

This form defines a Harish-Chandra subpair $(\mathrm{Symp}, \mathrm{Ham})$ of $(\mathrm{Aut}\,\mathcal{A}, \mathrm{Der}\,\mathcal{A})$.

DEFINITION: A **holomorphically symplectic structure** on X is a reduction X_{darb} of X_{coord} to $(\mathrm{Symp}, \mathrm{Ham})$.

A closed non-degenerate 2-form $\omega \in \Omega^2(X)$ restricts to a formal neighborhood of each point, defining a symplectic structure on each fiber of the jet bundle. The sheaf of trivializations of J identifying this symplectic form with the standard one is **locally nonempty** by the Darboux theorem. The jet of the form ω is a flat section of the jet bundle of 2-forms and hence the connection ∇_G has a solder form with coefficients in $\mathrm{Ham} \subset \mathrm{Der}\,\mathcal{A}$.

Quantizations as geometric structures

Consider the Weyl algebra

$$\mathcal{W} = \mathbb{C}[[z_1, \ldots, z_{2n}, \hbar]]/(z_i z_j - z_j z_i = \hbar \delta_{i,j+n}).$$

The pair $(\mathrm{Aut}\,\mathcal{W}, \mathrm{Der}\,\mathcal{W})$ is a Harish-Chandra pair with a morphism

$$(\mathrm{Aut}\,\mathcal{W}, \mathrm{Der}\,\mathcal{W}) \longrightarrow (\mathrm{Symp}, \mathrm{Ham})$$

given by reduction modulo $\hbar$.

THEOREM: (Bezrukavnikov-Kaledin-Nest-Tsygan) The set of isomorphism classes of deformation quantizations of (X, ω) is **in bijection** with the set of equivalence classes of reductions of X_{darb} to $(\mathrm{Aut}\,\mathcal{W}, \mathrm{Der}\,\mathcal{W})$.

Quantizations as geometric structures

THEOREM: (Bezrukavnikov-Kaledin-Nest-Tsygan) The set of isomorphism classes of deformation quantizations of (X, ω) is **in bijection** with the set of equivalence classes of reductions of X_{darb} to $(\mathrm{Aut}\,\mathcal{W}, \mathrm{Der}\,\mathcal{W})$.

PROOF: In one direction, for a reduction X_{quant} the sheaf of flat sections of $\mathrm{desc}_{X_{quant}}\mathcal{W}$ is a deformation quantizations.

In the other direction, suppose that $\mathcal{O}_\hbar$ is a deformation quantization of X. Then

$$J_\hbar := \lim_n \pi_{1*}(\mathcal{O}_X \widehat{\boxtimes} \mathcal{O}_\hbar / I^{n+1} \otimes_{\pi_2^{-1}\mathcal{O}_X} \pi_2^{-1}\mathcal{O}_\hbar),$$

is a flat bundle of algebras such that each fiber is a quantization of $(\mathcal{A}, \widehat{\omega})$. Its torsor of trivializations is locally nonempty since **formally locally quantizations are unique**.

Derivations of the Weyl algebra

On a formal disc every vector field preserving the symplectic form has a Hamiltonian, which is defined up to a constant. In other words, we have the following central extension of Lie algebras:

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{A} \longrightarrow \mathsf{Ham} \longrightarrow 0.$$

From the PBW theorem, the Weyl algebra as a vector space is isomorphic to $\mathcal{A}[[\hbar]]$. The projection modulo $\hbar$ is a Lie algebra map $\mathcal{W} \longrightarrow \mathcal{A}$, where the bracket on $\mathcal{W}$ is the algebraic commutator.

A Hochschild cohomology computation shows that **almost every derivation** of $\mathcal{W}$ is **inner**: concretely, there exist a central extension of Lie algebras:

$$0 \longrightarrow \mathbb{C}[[\hbar]] \longrightarrow \mathcal{W} \xrightarrow{a \mapsto \frac{1}{\hbar}\mathsf{ad}_a} \mathsf{Der}\, \mathcal{W} \longrightarrow 0.$$

Further reductions

Every vertical arrow in this commutative diagram is **not quite a deformation**:

$$\begin{array}{ccccc} (1, k[[\hbar]]) & \longrightarrow & (\mathrm{Aut}\,\mathcal{W}, \mathcal{W}) & \longrightarrow & (\mathrm{Aut}\,\mathcal{W}, \mathrm{Der}\,\mathcal{W}) \\ \downarrow & & \downarrow & & \downarrow \\ (1, k) & \longrightarrow & (\mathrm{Symp}, \mathcal{O}) & \longrightarrow & (\mathrm{Symp}, \mathrm{Ham}), \end{array}$$

PROBLEM: The groups $\mathrm{Aut}\,\mathcal{W}$ and Symp are different, so we did not obtain a deformation of Harish-Chandra pairs yet.

SOLUTION: Both $\mathrm{Aut}\,\mathcal{W}$ and Symp are prounipotent extensions of Sp.

A reduction of X_{coord} to $(GL, \mathrm{Der}\,\mathcal{A})$ is the same as the choice of isomorphism $J \approx \hat{\mathrm{S}}\Omega^{1,0}$, or the splitting of the natural filtration on the jet bundle. This is not always possible to do holomorphically, the obstruction being the so-called **Atiyah class**, but always possible to do **smoothly**.

Reduction of jets

Consider the graded algebra $\Omega^{*,*}\hat{\otimes}\hat{\mathsf{S}}\Omega^{1,0}$.

Let A_0 be the "fiberwise de Rham" derivation of tridegree $(1,0,-1)$ acting by

$$A_0(1 \otimes \alpha) = \alpha \otimes 1$$

and let K be the "fiberwise Koszul" derivation

$$K(\alpha \otimes 1) = 1 \otimes \alpha.$$

Their commutator $[A, K]$ acts on $\Omega^{p,q} \otimes \mathsf{S}^r\, \Omega^{1,0}$ by multiplication by $(p+r)$.

Let $\nabla := \nabla^{1,0} + \nabla^{0,1}$ be a smooth connection on $\Omega^{1,0}$, regarded as a derivation of our graded algebra.

DEFINITION: A connection ∇ is called **torsion-free** if $[\nabla, A_0] = 0$.

Reduction of jets

THEOREM: Let $A_1 := \nabla$ be a torsion-free connection. Define the $\Omega^{*,*}$-linear derivations $A_i, i \geq 2$ of $\Omega^{*,*}\widehat{\otimes}\hat{\mathsf{S}}\Omega^{1,0}$ by the formula

$$A_{n+1}(1 \otimes \alpha) := -\frac{1}{2(n+2)} \sum_{i=1}^{n} K[A_i, A_{n+i-1}](1 \otimes \alpha).$$

Then $D := \sum A_i$ **squares to zero**.

The operator D defines a **flat connection** on a bundle $\hat{\mathsf{S}}\Omega^{1,0}$. Its $D^{0,1}$ part determines a (non-standard) holomorphic structure on $\hat{\mathsf{S}}\Omega^{1,0}$, and $D^{1,0}$ is a holomorphic flat connection with respect to this holomorphic structure.

REMARK: If ∇ is the Levi-Civita connection of a Kähler metric, then $D^{1,0} = A_0 + \nabla^{1,0}$.

Reduction of jets

Let $X_{J-coord}$ be the $(GL, \mathsf{Der}\,\mathcal{A})$-torsor of trivializations of $(\hat{\mathsf{S}}\Omega^{1,0}, D^{0,1})$. Let $X_{?-coord}$ be the pushforward of $X_{J-coord}$ to $(\mathsf{Aut}\,\mathcal{A}, \mathsf{Der}\,\mathcal{A})$.

THEOREM: The torsor $X_{?-coord}$ is isomorphic to X_{coord}. In particular, the bundle $(\hat{\mathsf{S}}\Omega^{1,0}, D^{0,1})$ is isomorphic to the jet bundle $\mathsf{desc}_{X_{coord}}\mathcal{A}$ and $D^{1,0}$ to the Grothendieck flat connection ∇_G.

PROOF: $X_{?-coord}$ is **transitive**, due to locally $D = \nabla_G + higher\ order\ terms$.

THEOREM: For X Kähler, the complex $(\Omega^{0,*}, \overline{\partial})$ is **isomorphic** to the complex

$$(\Omega^{0,*}\hat{\otimes}\hat{\mathsf{S}}\Omega^{1,0}, D^{0,1}) \cap \mathsf{Ker}(D^{1,0})$$

via the map

$$\alpha \mapsto e^{-[K,\nabla^{1,0}]}\alpha.$$

This map could be thought of as the Taylor decomposition of α in holomorphic variables.

Reduction of symplectic jets

Suppose now X is Kähler and holomorphically symplectic. Then $\Omega^{1,0}$ and consequently $\hat{\mathrm{S}}\Omega^{1,0}$ naturally reduces to the structure group $Sp \subset GL$. Suppose, in addition, that ∇ preserves the form ω. Then the construction of the differential D returns the Ham-**valued** flat connection instead of just $\mathrm{Der}\,\mathcal{A}$-valued. Denote the corresponding (Sp, Ham)-torsor by X_{J-darb}. Consider the following commutative diagram of Harish-Chandra pairs:

$$\begin{array}{ccccc} (\mathrm{Aut}\,\mathcal{A}, \mathrm{Der}\,\mathcal{A}) & \longleftarrow & (\mathrm{Symp}, \mathrm{Ham}) & \longleftarrow & (\mathrm{Aut}\,\mathcal{W}, \mathrm{Der}\,\mathcal{W}) \\ \uparrow & & \uparrow & & \uparrow \\ (GL, \mathrm{Der}\,\mathcal{A}) & \longleftarrow & (Sp, \mathrm{Ham}) & \longleftarrow & (Sp, \mathrm{Der}\,\mathcal{W}). \end{array}$$

The set of equivalence classes of reductions of X_{J-darb} to $(Sp, \mathrm{Der}\,\mathcal{W})$ is in bijection to the set of equivalence classes of reductions of X_{darb} to $(\mathrm{Aut}\,\mathcal{W}, \mathrm{Der}\,\mathcal{W})$ and therefore to the set of isomorphism classes of quantizations of (X, ω).

The map $(Sp, \mathrm{Der}\,\mathcal{W}) \longrightarrow (Sp, \mathrm{Ham})$ **is a deformation!**

The curved algebra $\mathcal{QUA}$

Let $\mathcal{MW} \in T(Sp, \mathsf{Ham})$ be the Maurer-Cartan element defining the deformation $(Sp, \mathsf{Der}\,\mathcal{W}) \longrightarrow (Sp, \mathsf{Ham})$. Let $\mathsf{Prec}(\mathcal{MW})$ be the image of $\mathcal{MW}$ in $T(\mathsf{desc}_{X_{J-darb}}\mathsf{Ham}$. Denote by $\overline{\mathsf{desc}_{X_{J-darb}}\mathsf{Ham}}$ the corresponding deformation and by $\mathcal{QUA}$ the algebra $\hbar\overline{\mathsf{desc}_{X_{J-darb}}\mathsf{Ham}}$.

THEOREM: The set $\pi_0\mathcal{MC}(\mathcal{QUA})$ is in bijection with the set of isomorphism classes of quantizations.

Remind that for Kähler holomorphically symplectic X the algebra

$$\mathsf{desc}_{X_{J-darb}}\mathsf{Ham} = \mathsf{desc}_{X_{J-darb}}\mathcal{A}/\mathbb{C}$$

is isomorphic to

$$\Omega^{*,*}\hat{\otimes}\hat{\mathsf{S}}^{\geq 1}\Omega^{1,0}$$

with the differential

$$A_0 + \nabla^{1,0} + \overline{\partial} + \mathsf{ad}_R$$

where R is an element in

$$\mathsf{Hom}(\Omega^{1,0}, \Omega^{0,1}\hat{\otimes}\hat{\mathsf{S}}^{\geq 2}\Omega^{1,0}) = \Omega^{0,1}\hat{\otimes}\,\mathsf{Ham}^{\geq 2}.$$

The period map

Remind the diagram of central extensions and deformations

$$\begin{array}{ccccc} (1, \mathbb{C}[[\hbar]]) & \longrightarrow & (Sp, \mathcal{W}) & \longrightarrow & (Sp, \mathrm{Der}\,\mathcal{W}) \\ \downarrow & & \downarrow & & \downarrow \\ (1, \mathbb{C}) & \longrightarrow & (Sp, \mathcal{A}) & \longrightarrow & (Sp, \mathrm{Ham}). \end{array}$$

One associates a cohomology class to a central extension, measuring an obstruction to it being split:

$$c \in H^2(\mathrm{Ham}, Sp, k), \quad \overline{c} \in H^2(\mathrm{Der}\,\mathcal{W}, Sp, k[[\hbar]]).$$

A cocycle is a dg-morphism from a polynomial algebra to the Chevalley-Eilenber algebra, so we have two L_∞-maps

$$c : (Sp, \mathrm{Ham}) \longrightarrow (1, \mathbb{C}[1]), \quad \overline{c} : (Sp, \mathrm{Der}\,\mathcal{W}) \longrightarrow (1, \mathbb{C}[1][[\hbar]])$$

The period map, roughly speaking, is a descent of a deformation of the morphism c into the morphism $\overline{c}$. In our situation, it is more convenient to describe deformations of **ideals** instead of all morphisms.

Ideals

Let

$$0 \longrightarrow V \longrightarrow L \longrightarrow \mathfrak{h} \longrightarrow 0$$

be a central extension of Lie algebras. Take a splitting of vector spaces $\sigma : \mathfrak{h} \longrightarrow L$ and consider the 2-cocycle $c : \Lambda^2\mathfrak{h} \longrightarrow V$ given by

$$c(a,b) = [\sigma(a),\sigma(b)] - \sigma([a,b]).$$

Consider the dg-Lie algebra $\tilde{\mathfrak{h}}$ which is

$$\tilde{\mathfrak{h}} := \mathfrak{h} \oplus V \oplus V[1]$$

as a complex, with the differential given by $d(a,v,sw) = (0,w,0)$ and the bracket

$$[(a,v,sw),(a',v',sw')] = ([a,a'],c(a,a'),0).$$

LEMMA: Consider the maps $i_1(a) = (a,0,0)$ and $i_2(a,b) := (0,0,sc(a,b))$. Then (i_1,i_2) are Taylor components of an L_∞-map $\mathfrak{h} \longrightarrow \tilde{\mathfrak{h}}$.

DEFINITION: A subpair $(G, I) \subset (G, \mathfrak{h})$ is an **cocentral ideal** in an L_∞-algebra $\mathfrak{h}$ if the Chevalley-Eilenberg differential vanishes on $I^\perp \subset \mathfrak{h}^\vee \subset C^*(\mathfrak{h})$.

EXAMPLE: L is a cocentral ideal in $\tilde{\mathfrak{h}}$.

DEFINITION-THEOREM: Consider the dg-Lie subalgebra $T_I(G, \mathfrak{h})$ of $T(G, \mathfrak{h})$ consisting of derivations vanishing on $I^\perp$. Then $\pi_0 \mathcal{MC}(T_I(G, \mathfrak{h}))$ is in bijection with the set of isomorphism classes of **deformations of $(G, \mathfrak{h})$ with an ideal (G, I)**.

LEMMA: Let (P, α) be a $(G, \mathfrak{h})$-torsor over X. Then $\operatorname{desc} I$ is an ideal in $\operatorname{desc} \mathfrak{h}$. Moreover, the precomposition maps $T_I(G, \mathfrak{h})$ into $T_{\operatorname{desc} I}(\operatorname{desc} \mathfrak{h})$.

REMARK: The cohomology of the dg-Lie algebra $T_{\tilde{\mathfrak{h}}}(G, \mathfrak{h})$ is the relative cohomology $H(\mathfrak{h}, G, \tilde{\mathfrak{h}})$.

The period map

The existence of the diagram

$$\begin{array}{ccccc} (1,\mathbb{C}[[\hbar]]) & \longrightarrow & (Sp,\mathcal{W}) & \longrightarrow & (Sp,\mathrm{Der}\,\mathcal{W}) \\ \downarrow & & \downarrow & & \downarrow \\ (1,\mathbb{C}) & \longrightarrow & (Sp,\mathcal{A}) & \longrightarrow & (Sp,\mathrm{Ham}) \end{array}$$

implies that the element $\mathcal{MW}$ lies in $T_{(Sp,\mathcal{A})}(Sp,\mathrm{H\tilde{a}m})$.

Let X be a holomorphically symplectic manifold with a smooth connection preserving the symplectic form, and let X_{J-darb} be the corresponding (Sp,Ham)-torsor. Define by $X_{\tilde{J}-darb}$ its image under the morphism $(Sp,\mathrm{Ham}) \longrightarrow (Sp,\mathrm{H\tilde{a}m})$. Denote the deformation corresponding to element

$$\mathrm{Prec}(\mathcal{MW}) \in T_{\mathrm{desc}\,\mathcal{A}}(\mathrm{desc}\,\mathrm{H\tilde{a}m})$$

by $\mathcal{Q}\tilde{\mathcal{U}}\mathcal{A}$. By construction, the algebra $\mathcal{Q}\tilde{\mathcal{U}}\mathcal{A}$ is endowed by a (linear) L_∞-map into the trivial deformation of an abelian L_∞-algebra

$$\Omega^{*,*}[[\hbar]].$$

This is the L_∞ enhancement of the period map.

The period map

After this, the curved algebra $\mathcal{Q}\tilde{\mathcal{U}}\mathcal{A}$ could be safely forgotten, as the map $\mathcal{Q}\mathcal{U}\mathcal{A} \longrightarrow \mathcal{Q}\tilde{\mathcal{U}}\mathcal{A}$ which in concrete terms look like

$$\Omega^{*,*}\hat{\otimes}\,\mathrm{Der}\,\mathcal{W} \longrightarrow \Omega^{*,*}\hat{\otimes}\tilde{\mathrm{Der}}\mathcal{W}$$

with components

$$f_1(\beta\otimes v) = (\beta\otimes v, 0, s\bar{c}(\beta\otimes v, \iota(\alpha))),$$
$$f_2(\beta\otimes v, \gamma\otimes w) = (0, 0, s\bar{c}(\beta\otimes v, \gamma\otimes w))$$

is a L_∞ map which is quasiisomorphism modulo $\hbar$, and hence induces a bijection $\pi_0\mathcal{MC}(\mathcal{Q}\mathcal{U}\mathcal{A}) \longrightarrow \pi_0\mathcal{MC}(\mathcal{Q}\tilde{\mathcal{U}}\mathcal{A})$.

DEFINITION: The L_∞-map $\mathcal{Q}\mathcal{U}\mathcal{A} \longrightarrow \Omega^{*,*}[1][[\hbar]]$ given by

$$\mathcal{P}_1(\beta\otimes v) = \bar{c}(\beta\otimes v, \iota(\alpha)),$$
$$\mathcal{P}_2(\beta\otimes v, \gamma\otimes w) = \bar{c}(\beta\otimes v, \gamma\otimes w)$$

is called the **period map**.

How to classify quantizations

THEOREM: (Katzarkov-Kontsevich-Pantev) Suppose L and M are two curved dg-Lie algebras over $k[[\hbar]]$ that are topologically free as $k[[\hbar]]$-modules. with curvature elements divisible by $\hbar$, and let $f : C(L) \longrightarrow C(M)$ be a morphism of dg-coalgebras. Its linear part induces a **map of complexes** $\mathrm{gr}_\hbar f_1 : \mathrm{gr}_\hbar L \longrightarrow \mathrm{gr}_\hbar M$. Suppose that M is $\hbar$-filteredly quasiisomorphic to an abelian Lie algebra and suppose that $\mathrm{gr}_\hbar f_1$ induces an injection on cohomology. Then **L is homotopy abelian as well**.

COROLLARY: In the situation above, $\pi_0\mathcal{MC}(L) = H(\mathrm{gr}_\hbar L)$.

The map $H((\mathcal{P}_\hbar)_1)$ could be identified with the standard embedding $H^1(X, \mathrm{Ham}) = \mathbb{H}^1(X, \Omega^{\geq 1}_{dR}) \longrightarrow \mathbb{H}^2(X, \Omega^*_{dR})$.

This reproves the theorem of Nest and Tsygan.

Calculation of $\mathcal{QUA}$

Let P be the bundle of Sp-frames on X and let A be the solder form in $\Omega^1(P)\hat{\otimes}\,\mathrm{Ham}$ of the connection $A_0 + \nabla^{1,0} + \overline{\partial} + \mathrm{ad}_R$. Then the algebra $\mathcal{QUA}$ is the algebra

$$\Gamma(\Omega^*(P)\hat{\otimes}\,\mathrm{Der}\,\mathcal{W}) = \Omega^*(X)\hat{\otimes}\,\mathrm{Der}\,\mathcal{W}\Omega^{1,0}$$

with the differential

$$d + \mathrm{ad}_{\iota(A)}$$

and the curvature

$$H := d\iota(A) + \frac{1}{2}[\iota(A), \iota(A)].$$

Here $\iota : \mathrm{Ham} = \mathrm{Ham}\otimes 1 \longrightarrow \mathrm{Ham}[[\hbar]] = \mathrm{Der}\,\mathcal{W}$ is the embedding.

Moyal-Weyl deformation

The **Moyal-Weyl bracket** is better to describe for the central extensions. Under the identification $\mathcal{A}[[\hbar]] \approx \mathcal{W}$, the commutator in $\mathcal{W}$ of two elements $f, g \in \mathcal{A}$ is equal to

$$[f, g] = [e^{\frac{1}{2}\hbar\pi}(f \otimes g) - e^{\frac{1}{2}\hbar\pi}(g \otimes f)] = e^{\frac{1}{2}\hbar\pi}(f \wedge g).$$

Note that when one of the f or g is in the $\mathrm{Sym}^{\leq 2}$, the **Moyal-Weyl commutator is equal to the Poisson bracket** (times $\hbar$). Indeed, $[f, g] = \hbar\{f, g\}$ plus summands involving third or higher derivatives of f and g.

COROLLARY: The algebra $\mathcal{QUA}$ is isomorphic to $\Omega^*(X) \hat{\otimes} \operatorname{Der} \mathcal{W}$ with the differential

$$A_0 + \nabla^{1,0} + \overline{\partial} + \mathrm{ad}_{\iota(R)}.$$

In particular, the curvature **is a $(0, 2)$-form**.

One more time about characteristic classes

Let (P, α) be a (Sp, Ham)-torsor over X. Consider the dg-Lie algebra $T_{\mathcal{A}}(Sp, \mathrm{Ham})$.

LEMMA: $H(T_{\mathcal{A}}(Sp, \mathrm{Ham})) = H(\mathrm{Ham}, Sp, \mathcal{A}) = H(\mathrm{Ham}_0, Sp)$, the relative cohomology of the Lie algebra of Hamiltonian vector fields **preserving a point**.

PROOF: As a (Ham, Sp)-module, $\mathcal{A} = \mathrm{Hom}_{U\,\mathrm{Ham}_0}(U\,\mathrm{Ham}, \mathbb{C})$. The lemma follows from Shapiro lemma.

DEFINITION: The characteristic map

$$T_{\mathcal{A}}(\mathrm{Ham}, Sp) = C^*(\mathrm{Ham}, Sp, \mathcal{A}) \longrightarrow \mathrm{desc}_{X_{J-coord}} \mathcal{A} = \Gamma(\Omega^*(P) \hat{\otimes} \mathcal{A})$$

in this situation is called the **Rozansky-Witten map** RW.

LEMMA: Consider the precomposition map

$$\mathrm{Prec} : T_{\mathcal{A}}(\mathrm{Ham}, Sp) \longrightarrow T_{\mathrm{desc}\,\mathcal{A}}(\mathrm{desc}\,\mathrm{Ham}).$$

Then

$$\frac{1}{n!} \mathrm{Prec}(F)(\alpha, \alpha, \dots, \alpha) = RW(F).$$

Rozansky-Witten classes as curvature

The algebra

$$\mathcal{QUA} = \Omega^{*,*}(X)\hat{\otimes}\hat{\mathsf{S}}^{\geq 1}\Omega^{1,0}[[\hbar]]$$

comes equipped with the morphism from the algebra

$$\overline{\mathrm{desc}\,\mathcal{A}} = \Omega^{*,*}(X)\hat{\otimes}\hat{\mathsf{S}}\Omega^{1,0}[[\hbar]]$$

with the differential given by a formula that looks almost the same:

$$A_0 + \nabla^{1,0} + \overline{\partial} + \iota(R)\cdot .$$

The curvature of $\overline{\mathrm{desc}\,\mathcal{A}}$ is the **Rozansky-Witten invariant** associated to the element $\mathcal{MW}$. The curvature of $\mathcal{QUA}$ is, correspondingly, its image.

REMARK: In the description of RW-invariants in terms of graphs, the class $RW(\mathcal{MW})$ corresponds to the graps

$$\sum_{k\geq 1}\frac{\hbar^{2k}}{(2k+1)!}\Theta_{2k+1},$$

where Θ_{2k+1} is a graph with two vertices and $2k+1$ edges between them.

Smaller model for $\mathcal{QUA}$

OBSERVATION: Suppose that $(L^{*,*}, d^{1,0}+d^{0,1}, h^{0,2})$ is a bigraded curved dg-Lie algebra such that the curvature has the degree $(0,2)$. Then the space of $d^{1,0}$-closed elements of bidegrees $(0,*)$ together with the differential $d^{0,1}$ is a curved dg-Lie subalgebra.

REMIND:

$$\Omega^{0,*}(X)\hat{\otimes}\hat{\mathsf{S}}\Omega^{1,0} \cap \mathsf{Ker}(A_0 + \nabla) = \Omega^{0,*}$$

with the isomorphism given by $e^{-[K,\nabla^{1,0}]}$.

COROLLARY: The complex $(\Omega^{0,*}(X)[[\hbar]], \overline{\partial})$ has the structure of a curved dg-Lie algebra $\hbar$-filteredly quasiisomorphic to $\overline{\mathrm{desc}\,\mathcal{A}}$.

COROLLARY: The complex $(\Omega^{1,*}_{\partial-closed}(X)[[\hbar]], \overline{\partial})$ has the structure of a curved dg-Lie algebra $\hbar$-filteredly quasiisomorphic to $\mathcal{QUA}$. We will call this algebra $\mathcal{QUF}$.

Even smaller model for $\mathcal{QUF}$

Suppose now that X is compact, Kähler and that its Levi-Civita connection preserves the holomorphically symplectic structure. In this case X is automatically **hyperKähler**. The algebra $\mathcal{QUF}$ is

$$(\Omega^{1,*}_{\partial-closed}(X)[[\hbar]], \overline{\partial} + d_{\hbar}..., H).$$

THEOREM: (Homotopy transfer) There exists a curved dg-Lie algebra stucture $\mathcal{QUH}$ on the complex

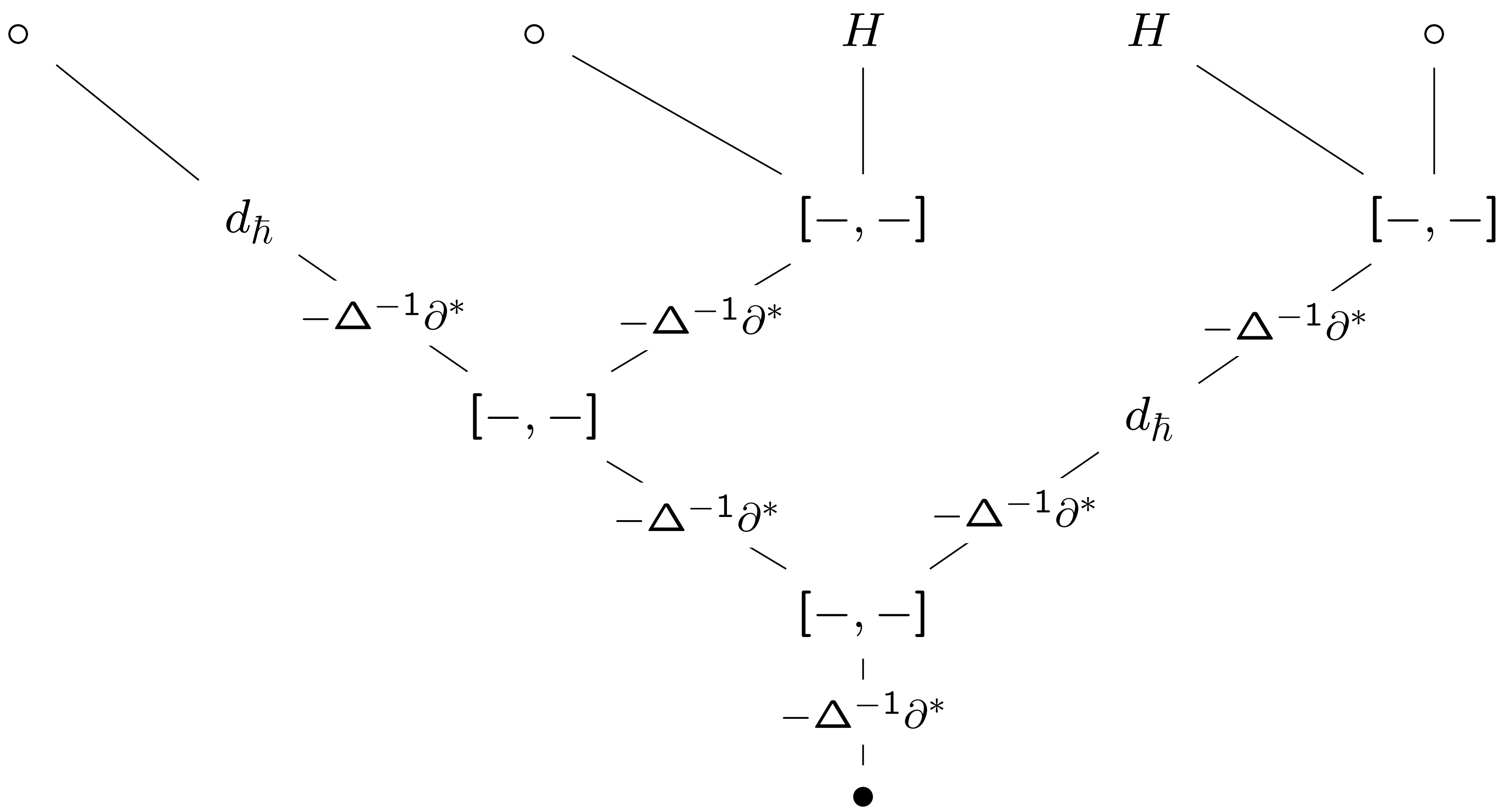
$$(H^{\geq 1,*}(X)[[\hbar]], \tilde{d}, \tilde{H})$$

such that $\tilde{d} = 0$ modulo $\hbar$ together with an **explicitly given** $\hbar$-filtered quasiisomorphism $\mathcal{QUH} \longrightarrow \mathcal{QUF}$.

A posteriori, the operators $\tilde{d}$ and $\tilde{H}$ **vanish**.

As a corollary, $\pi_0\mathcal{MC}(\mathcal{QUH}) = H^{\geq 1,*}(X)[[\hbar]]$.

Sum over trees



conilpotent Lie coalgebras

An unrelated results concerns the cohomology algebras of **conilpotent Lie coalgebras**.

DEFINITION: A coalgebra L is called **conilpotent** if its dual profinite Lie algebra $L^\vee$ is a projective limit of finite dimensional nilpotent Lie algebras. Alternativelty, it means that there exists an increasing ascending exhaustive filtration by subspaces F_*L such that $F_0L = 0$ and such that $\Delta(F_i)$ lies inside $\sum_{p+q=i} F_p \otimes F_q \subset L \otimes L$.

The tensor powers of conilpotent Lie coalgebra L inherit the filtration and hence its Chevalley-Eilenberg algebra $C = C(L)$ is then also endowed with an ascending exhaustive filtration such that $F_0C = k = C^0$. The Chevalley-Eilenberg differential preserves this filtration.

Bar construction

Let A be an associative dg-algebra. Its **bar-construction** is a coassociative dg-coalgebra $\mathsf{B}A$ which, as a coalgebra, is the tensor coalgebra generated by $A[1]$, and whose codifferential is the unique codderivation of the tensor coalgebra whose corestriction to $A[1]$ is given by the differential and multiplication in A. Modulo signs, we have

$$d_{\mathsf{B}}(a_1 \otimes \cdots \otimes a_n) = \sum \pm a_1 \otimes \ldots da_k \cdots \otimes a_n + \sum \pm a_1 \otimes \ldots a_i a_{i+1} \cdots \otimes a_n.$$

DEFINITION: An **A_∞-algebra** A is a graded vector space with a square-zero coderivation $d_{\mathsf{B}}A$ on the coaugmented tensor coalgebra on $A[1]$. An **A_∞-morphism** is a dg-coalgebra morphism between Bar-constructions.

Filtrations

For an A_∞-algebra A the bar-construction $\mathsf{B}A$ is a filtered coalgebra. The filtration is given by

$$F_i \mathsf{B}A := \bigoplus_{k=0}^{i} (A[1])^{\otimes k}.$$

If an A_∞-algebra A is determined by Taylor components $(m_1, m_2, \dots)$, then $\mathrm{gr}_F\, \mathsf{B}A$ is $\oplus_i A[1]^{\otimes i}$ with the differential m_1. In particular, $m_1^2 = 0$, and it defines a structure of a complex on A.

Note that if $f : \mathsf{B}A \longrightarrow \mathsf{B}B$ is a morphism of dg-coalgebras, then it automatically preserves this filtration. If the Taylor components of f are $(f_1, f_2, \dots)$, then $\mathrm{gr}_F\, f = f_1$. In particular, $f_1 m_{A,1} = m_{B,1} f_1$.

An A_∞-morphism $f : \mathsf{B}A \longrightarrow \mathsf{B}B$ is called a **quasiisomorphism** if it is a filtered quasiisomorphism of dg-coalgebras

Minimal models

DEFINITION: An A_∞-algebra A is called **minimal** if $\mathrm{gr}_F\,\mathrm{B}A$ has vanishing differential.

THEOREM: For any A_∞-algebra A there exists a **unique up to A_∞-isomorphism** minimal A_∞-algebra H quasiisomorphic to A.

1-generatedness

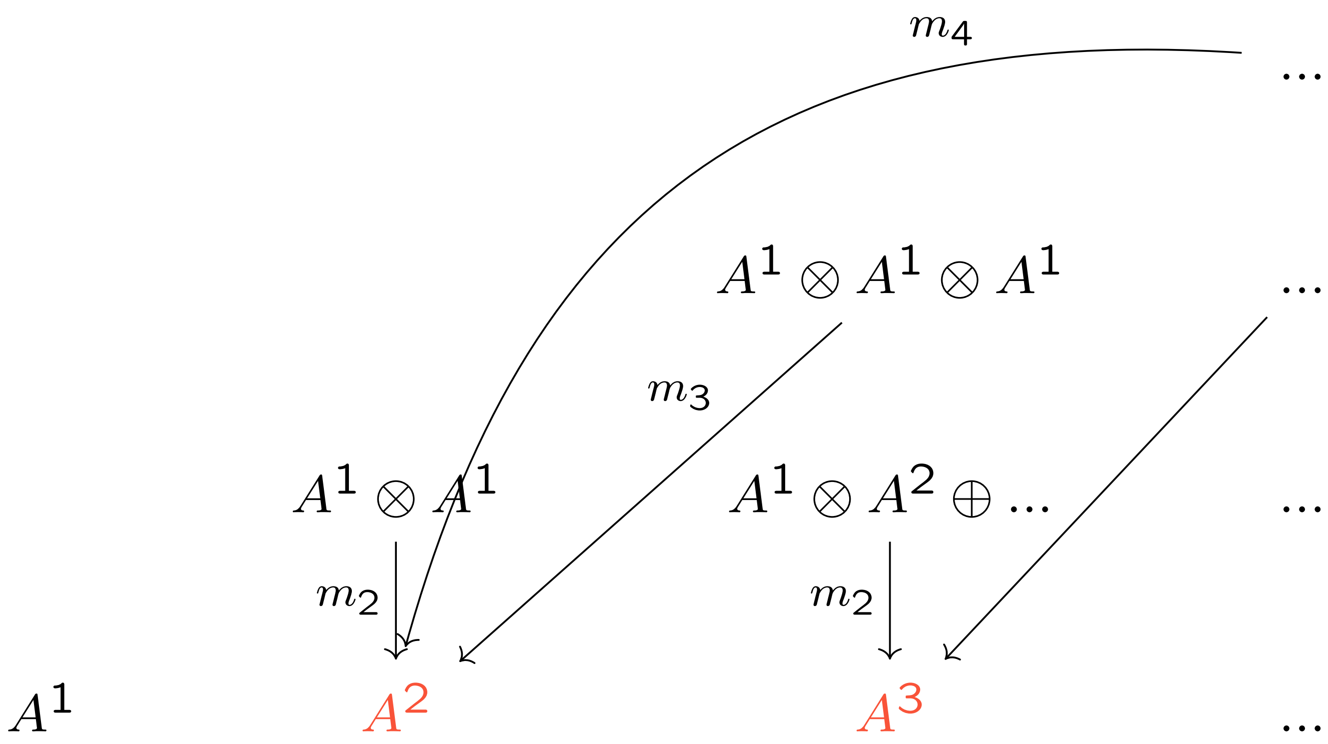
Let $(A, m_2, m_3, \dots)$ be a minimal A_∞-algebra. We are interested in the cohomology of its bar-complex $H(\mathrm{B}A)$. As $\mathrm{B}A$ is a filtered (by the tensor filtration) complex, its cohomology $H(\mathrm{B}A)$ inherit the filtration. That is, $F^k H(\mathrm{B}A)$ are classes that could be represented by cocycles that lie in $F^k \mathrm{B}A$.

DEFINITION: A minimal A_∞-algebra A is called **1-generated** if it is positively graded and $F^1 H^j(\mathrm{B}A) = 0$ for $j \geq 1$.

LEMMA: A positively graded A_∞ algebra A is 1-generated if and only if any element $x \in A^k$ can be expressed as a linear combination of iterated compositions of maps of the form $\mathrm{Id}^{\otimes i} \otimes m_j \otimes \mathrm{Id}^{\otimes k}$ applied to elements in $A^{\otimes n}$.

Proof of the lemma

Induction on n.



Bar-Cobar vanishing

LEMMA: or a conilpotent Lie coalgebra L, the cohomology of $\mathsf{B}C^{>0}(L)$ vanishes in positive degrees. The coalgebra $H^0(\mathsf{B}C^{>0}(L))$ is isomorphic to $U(L)$, the conilpotent coenveloping coalgebra of L.

PROOF: We have two naturally defined filtrations on $\mathsf{B}C^{>0}(L)$. The "stupid" filtration on $C^{>0}(L)$ given by $G^k = \oplus_{i \geq k} C^i(L)$ extends to a filtration on $\mathsf{B}C^{>0}(L)$, which we will denote by G. The filtration induced from the conilpotent filtration on L will be denoted by N. The filtration G is descending and non-complete, the filtration N is ascending and exhaustive.

Consider the complex $\mathrm{gr}_N \mathsf{B}C^{>0}(L)$. The filtration G induces a filtration on it, which we will also denote by G. An important fact is that G is finite on each N_i/N_{i-1}. Consider now the complex $\mathrm{gr}_G \mathrm{gr}_N \mathsf{B}C_{>0}(L)$. We want to show that its higher cohomology vanish. From finiteness of F and exhaustiveness of N it will follow that higher cohomology of $\mathsf{B}C^{>0}(L)$ would vanish as well.

Bar-Cobar vanishing

$$
\begin{array}{ccccccc}
 & & & & & & \cdots \\
 & & & & C^1 \otimes C^1 \otimes C^1 & \longrightarrow & \cdots \\
 & & & & \downarrow & & \\
 & & C^1 \otimes C^1 & \longrightarrow & C^1 \otimes C^2 \oplus \ldots & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
C^1 & \longrightarrow & C^2 & \longrightarrow & C^3 & \longrightarrow & \cdots
\end{array}
$$

Bar-Cobar vanishing

$$
\begin{array}{cccc}
 & & & \dots \\
 & & S^3C^1 & \dots \\
 & S^2C^1 & 0 & \dots \\
C^1 & 0 & 0 & \dots
\end{array}
$$

COROLLARY: Let L be a conilpotent Lie coalgebra. Let $H(L)$ be a minimal A_∞-algebra quasiisomorphic to $C(L)$. Then H is 1-generated.

THANK YOU FOR YOUR ATTENTION!